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学术论文

Robust Invariant Sets Generation for State-Constrained Perturbed (Polynomial) Systems
状态约束不确定系统的不变集估计

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Main Contribution: describe the maximal robust invariant set as the zero level set of the unique Lipschitz continuous viscosity solution to a Hamilton-Jacobi equation

1. Problem Description

The state-constrained perturbed dynamical system of interest is of the following form:

$$\dot{x}(t) = f(x(t), d(t)) \quad (1)$$

where $x(\cdot): [0, \infty) \to X$, $d(\cdot): [0, \infty) \to D$, $D = \{d \in R^m \mid \wedge_{i=1}^{n_d} h_i^D(d) \le 0\}$ is a compact set in R^m, $X = \{x \in R^n \mid \wedge_{i=1}^{n_X} h_i(x) \le 0\}$ is a compact set in R^n.

Remark 1 $d(t)$ is often used to incorporate model uncertainties and external disturbances.
Remark 2 Thanks to Kirszbraun's theorem, $f(x, d)$ can be any continuous function over $x \in R^n$ and $d \in R^m$, and locally Lipschitz-continuous over x uniformly over d.

Definition 1 A perturbation policy, denoted by π, refers to a measurable function $d(\cdot): [0, \infty) \to D$. The set of all perturbation policies is denoted by $\mathcal{D}$.

Denote the trajectory of system (1) initialized at $x_0 \in X$ and subject to perturbation $\pi \in \mathcal{D}$ by $\phi_{x_0}^{\pi}(t)$:
Definition 2 The maximal robust invariant set $\mathcal{R}_0$ is the set of states such that every possible trajectory of system (1) starting from it never leave X, i.e.

$$\mathcal{R}_0 = \{x \mid \phi_x^{\pi}(t) \in X, \forall t \in [0, \infty), \forall \pi \in \mathcal{D}\}.$$

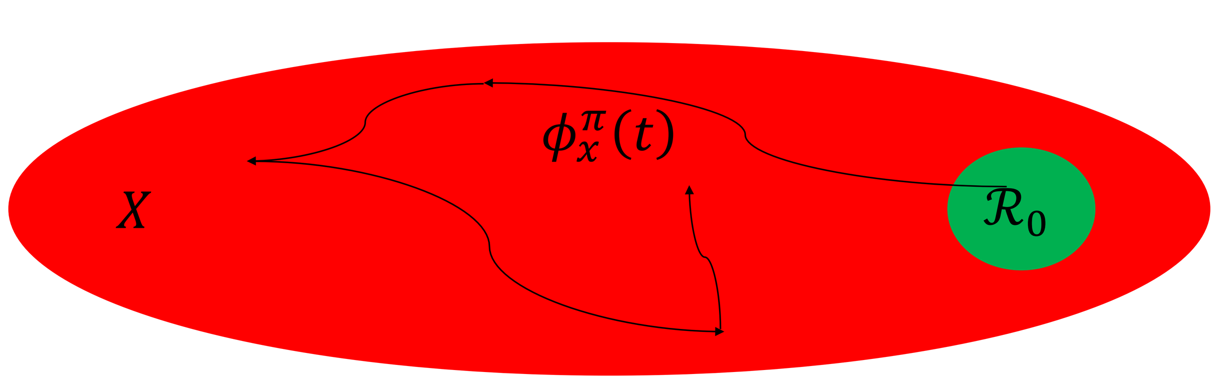

Fig. 1 An illustration of the maximal robust invariant set

2. Background

- A fundamental problem in the theory of dynamical systems is the computation of robust invariant sets, with applications ranging from systems analysis over controller design to safety verification.
- Synthesizing robust invariant sets has been the subject of extensive research over the past several decades, resulting in the emergence of a number of theories and corresponding computational approaches, e.g., Lyapunov function-based methods, fixed-point methods and viability theory and so on.
- ***There is no previous work on the use of Hamilton-Jacobi equations having continuous viscosity solutions to characterize the maximal robust invariant set for state-constrained perturbed dynamical systems.***

✓ The continuity and uniqueness property of the viscosity solution facilitates the use of well-developed numerical methods to solve the Hamilton-Jacobi equations.

3. Hamilton-Jacobi Equations

Let $h_j'(x) = \frac{h_j(x)}{1+h_j^2(x)}$, $j = 1, \dots, n_X$. Define the value function:

$$V(x) := \sup_{\pi \in \mathcal{D}} \sup_{t \in [0,\infty)} \max_{j \in \{1,\dots,n_X\}} \{e^{-\alpha t} h_j(\phi_x^{\pi}(t))\}, \alpha \in [0, \infty).$$

Theorem 1 $\mathcal{R}_0 = \{x \in R^n \mid V(x) \le 0\}$. Especially, when $\alpha > 0$, $\mathcal{R}_0 = \{x \in R^n \mid V(x) = 0\}$.

Lemma 1 If $\alpha > 0$, $V(x)$ is locally Lipschitz continuous over R^n. If $\alpha = 0$, $V(x)$ is lower semi-continuous over R^n.

Theorem 2 If $\alpha > 0$, $V(x)$ is the unique bounded and Lipschitz-continuous viscosity solution to the Hamilton-Jacobi partial differential equation (2):

$$\min\left\{\inf_{d \in D}\left(\alpha V(x) - \frac{\partial V}{\partial x} f(x, d)\right), V(x) - \max_{j \in \{1,\dots,n_X\}} h_j'(x)\right\} = 0 \quad (2)$$

Remark 3 If $\alpha = 0$, $V(x)$ is the minimal lower semi-continuous viscosity solution to (2).
Remark 4 If $V(x), f(x, d), h_j(x), j = 1, \dots, n_X$, are polynomials, solving (2) with $\alpha = 0$ can be relaxed as a convex programming problem. For details, please refer to [1].

4. Experiment

Consider a two-dimensional system, corresponding to a Moore-Greitzer model of a jet engine with the controller $u = 0.8076x - 0.9424y$,

$$\begin{cases} \dot{x} = -y - \frac{3}{2}x^2 - \frac{1}{2}x^3 + d \\ \dot{y} = u, \end{cases}$$

where $X = \{(x, y) \mid x^2 + y^2 - 0.04 \le 0\}$ and D= $\{d \mid d^2 - 0.02^2 \le 0\}$.

The estimated maximal robust invariant set, which is computed by solving (2) with α = 1, is displayed in Fig. 2. The level sets of the corresponding computed viscosity solution are shown in Fig. 2.

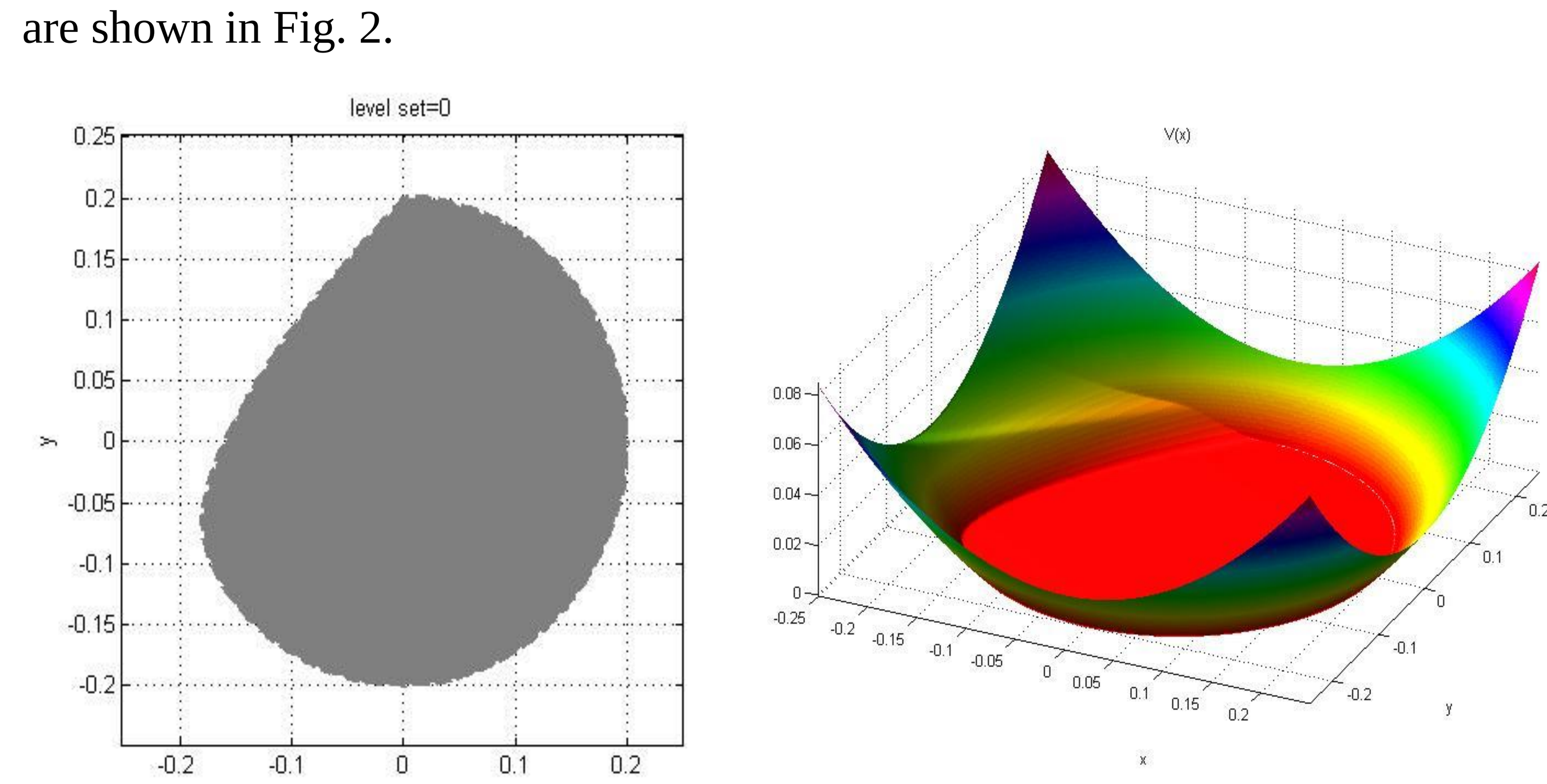

Fig. 2. Left: Gray region is an estimate of the maximal robust invariant set obtained by numerically solving (2) with α = 1. Right: Level sets of the computed viscosity solution to (2) with α = 1.

5. Our Related Work

[1]Bai Xue, Qiuye Wang, Naijun Zhan, Martin Fränzle. Robust invariant sets generation for state-constrained perturbed polynomial systems. HSCC 2019: 128-137.
[2]Bai Xue, Qiuye Wang, Naijun Zhan, Martin Fränzle. Reach-Avoid Differential Games Based on Invariant Generation. arXiv:1811.03215.
[3]Bai Xue, Martin Fränzle, Naijun Zhan. Inner-Approximating Reachable Sets for Polynomial Systems with Time-Varying Uncertainties. arXiv:1811.01086.
[4]Bai Xue, Qiuye Wang, Naijun Zhan, Shijie Wang, Zhikun She. Synthesizing Robust Domains of Attraction for State-Constrained Perturbed Polynomial Systems. arXiv:1812.10588.
[5]Bai Xue, Naijun Zhan, Yangjia Li. Regions of Attraction Generation for State-Constrained Perturbed Discrete-Time Polynomial Systems. arXiv:1810.11767.